LECTURE: 3-6 DERIVATIVES OF LOGARITHMIC FUNCTIONS

Review: Derivatives of Exponential Functions:

- $\frac{d}{dx}e^x = $ e^x
- $\frac{d}{dx}a^x = $ $\ln a \cdot a^x$

Example 1: Find a formula for the derivatives of the following functions.

(a) $y = \ln x$

$e^y = e^{\ln x}$

$e^y = x$

$e^y \frac{dy}{dx} = 1$

$\boxed{\frac{dy}{dx} = 1/e^y = 1/x}$

(b) $y = \log_b x$

$b^y = b^{\log_b x}$

$b^y = x$

$b^y \ln b \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{\ln b \cdot b^y} = \boxed{\frac{1}{\ln b x}}$

Derivatives of Logarithmic Functions:

- $\frac{d}{dx}\ln x = $ $\boxed{1/x}$
- $\frac{d}{dx}\log_b x = $ $\boxed{\frac{1}{(\ln b)x}}$

Example 2: Find derivatives of the following functions.

(a) $y = \ln(4x^2 + 5)$

$y' = \frac{1}{4x^2+5} \cdot \frac{d}{dx}(4x^2+5)$

$= \frac{1}{4x^2+5} \cdot 8x$

$= \boxed{\frac{8x}{4x^2+5}}$

(b) $y = \ln(\tan x)$

$y' = \frac{1}{\tan x} \cdot \frac{d}{dx} \tan x$

$= \boxed{\frac{\sec^2 x}{\tan x}}$

Example 3: Find derivatives of the following functions.

(a) $f(x) = \log_{10} \sqrt{x}$

$$f'(x) = \frac{1}{\ln 10 \sqrt{x}} \cdot \frac{d}{dx}\sqrt{x}$$
$$= \frac{1}{\ln 10 \sqrt{x}} \cdot \frac{1}{2} x^{-1/2}$$
$$= \frac{1}{\ln 10 \sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \boxed{\frac{1}{2\ln 10 x}}$$

(b) $g(x) = \log_2(\cos x)$

$$g'(x) = \frac{1}{\ln 2 \cos x} \cdot (-\sin x)$$
$$= \boxed{-\frac{\tan x}{\ln 2}}$$

Example 4: Differentiate f and find the domain of f'.

(a) $f(x) = \sqrt{5 + \ln x}$ $= (5+\ln x)^{1/2}$

$$f'(x) = \frac{1}{2}(5+\ln x)^{-1/2} \cdot \frac{d}{dx}(5+\ln x)$$
$$= \frac{1}{2\sqrt{5+\ln x}} \cdot \frac{1}{x}$$
$$= \boxed{\frac{1}{2x\sqrt{5+\ln x}}}$$

need $x > 0$ for $\ln x$ and $5+\ln x > 0$

$\ln x > -5$

$\boxed{x > e^{-5}}$

(b) $f(x) = \dfrac{x}{1 - \ln(x+1)}$

$$f'(x) = \left(\frac{1-\ln(x+1) - x\cdot\left(\frac{-1}{x+1}\right)}{(1-\ln(x+1))^2}\right)\frac{\frac{x+1}{1}}{\frac{x+1}{1}}$$
$$= \frac{x+1-(x+1)\ln(x+1)+x}{(x+1)(1-\ln(x+1))^2}$$
$$= \boxed{\frac{2x+1 - x\ln(x+1) - \ln(x+1)}{(x+1)(1-\ln(x+1))^2}}$$

$x \neq -1$ $\quad x+1 > 0 \Rightarrow x > -1$

also $1-\ln(x+1) \neq 0$

$1 \neq \ln(x+1)$

$e \neq x+1$, so $x \neq e-1$

$\boxed{D: (-1, e-1) \cup (e-1, \infty)}$

Example 5: Differentiate the following functions.

(a) $y = \ln|x|$.

Case 1

if $x > 0$ (positive) then $|x| = x$

and $y = \ln|x| = \ln x$; $y' = 1/x$

Case 2

if $x < 0$ (neg) then $|x| = -x$

and $y = \ln|x| = \ln(-x)$; $y' = \frac{1}{-x}(-1) = 1/x$

so if $y = \ln|x|$;

$\boxed{y' = 1/x \text{ w/no abs!}}$

(b) $f(x) = \ln|\sec x + \tan x|$

$$f'(x) = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$
$$= \boxed{\sec x}$$

It is often easier to first use the rules of logarithms to expand a logarithmic expression before taking the derivative. To do this properly you first must recognize when these rules can be applied and apply them correctly.

Rules and Non-Rules for Logarithms

- $\ln(AB) =$ $\ln A + \ln B$
- $\ln(A/B) =$ $\ln A - \ln B$
- $\ln(A^r) =$ $r \ln A$
- $\ln(A + B) =$ No rule $\neq \ln A + \ln B$
- $\ln(A - B) =$ No rule $\neq \ln A - \ln B$
- $(\ln A)^r =$ No rule $\neq r \ln A$

Example 6: Differentiate the following functions by first expanding the expressions using the rules for logarithms. Explain *why* this is the better way to proceed in each case.

(a) $f(x) = \ln\sqrt{5x+2}$

$= \ln(5x+2)^{1/2}$

$= \frac{1}{2}\ln(5x+2)$

$f'(x) = \frac{1}{2}\cdot\left(\frac{1}{5x+2}\right)\cdot 5$

$= \boxed{\frac{5}{2(5x+2)}}$

would have had to chain twice w/out expansion

(b) $g(x) = \log_5(x^2\sqrt{x+1})$

$= 2\log_5 x + \frac{1}{2}\log_5(x+1)$

$g'(x) = \frac{2}{(\ln 5)x} + \frac{1}{2}\frac{1}{(\ln 5)(x+1)}$

$= \frac{2}{(\ln 5)x}\frac{2(x+1)}{2(x+1)} + \frac{1}{2}\frac{1}{(\ln 5)(x+1)}\frac{x}{x}$

$= \frac{4x+4+x}{2\ln 5\, x(x+1)}$

$= \boxed{\frac{5x+4}{2\ln 5\, x(x+1)}}$

Example 7: Differentiate $f(x) = \ln\left(\frac{x(x^2+1)^2}{\sqrt{2x^4-5}}\right)$

$f(x) = \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^4-5)$

$f'(x) = \frac{1}{x} + \frac{2}{x^2+1}\cdot 2x - \frac{1}{2}\left(\frac{1}{2x^4-5}\right)\cdot 8x^3$

$= \boxed{\frac{1}{x} + \frac{4x}{x^2+1} - \frac{4x^3}{2x^4-5}}$

Example 8: Differentiate the following functions.

(a) $f(x) = (\ln x)^5$

$f'(x) = 5(\ln x)^4 \cdot 1/x$

$= \boxed{\dfrac{5(\ln x)^4}{x}}$

(b) $f(x) = \ln(x^5)$

w/out expansion:

$f'(x) = \dfrac{1}{x^5} \cdot \dfrac{d}{dx} x^5$

$= \dfrac{1}{x^5} \cdot 5x^4$

$= \boxed{\dfrac{5}{x}}$

w/ expansion

$f(x) = 5 \ln x$

$f'(x) = 5 \cdot (1/x)$

$= \boxed{5/x}$

Logarithmic Differentiation

Finding derivatives of complicated functions involving products, quotients and powers can often be simplified using logarithms. This technique is called logarithmic differentiation.

Example 9: Find the derivative of $y = \dfrac{x^7\sqrt{x^3+1}}{(5x+1)^4}$.

to find y' you need to use the quotient, product + chain rules!

* log both sides + expand!

$\ln y = \ln\left(\dfrac{x^7\sqrt{x^3+1}}{(5x+1)^4}\right)$

$\ln y = 7\ln x + \frac{1}{2}\ln(x^3+1) - 4\ln(5x+1)$

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{7}{x} + \dfrac{1}{2}\dfrac{1}{(x^3+1)}3x^2 - \dfrac{4}{5x+1}\cdot 5$ ← solve for dy/dx

$\dfrac{dy}{dx} = \left(\dfrac{7}{x} + \dfrac{3x^2}{2(x^3+1)} - \dfrac{20}{5x+1}\right)y$ ← input y

$$\boxed{\dfrac{dy}{dx} = \left(\dfrac{7}{x} + \dfrac{3x^2}{2(x^3+1)} - \dfrac{20}{5x+1}\right)\left(\dfrac{x^7\sqrt{x^3+1}}{(5x+1)^4}\right)}$$

If you don't input y, your problem is wrong / incomplete.

Derivative Rules: Let n and a be constants. (Note, there is no rule when there is a variable in the base *and* the exponent.)

- $\frac{d}{dx}x^n =$ $n\,x^{n-1}$

  here the exponent is constant

- $\frac{d}{dx}a^x =$ $(\ln a)\,a^x$

  here the base is constant

When you have a variable in both the base and the exponent you **must** use Logrithmic differentiation to find the derivative of the function.

Example 10: Find the derivatives of the following functions using logarithmic differentiation.

(a) $y = x^{2/x}$

$$\ln y = \ln x^{2/x}$$

$$\ln y = \frac{2}{x} \cdot \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = -2x^{-2}\ln x + \frac{2}{x}\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-2\ln x}{x^2} + \frac{2}{x^2}$$

$$\frac{dy}{dx} = \left(\frac{2-2\ln x}{x^2}\right)y$$

recall, $y = x^{2/x}$

$$\boxed{\frac{dy}{dx} = \left(\frac{2-2\ln x}{x^2}\right)\cdot x^{2/x}}$$

(b) $y = (\ln x)^{\cos x}$

$$\ln y = \ln(\ln x)^{\cos x}$$

$$\ln y = \cos x \cdot \ln(\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \cdot \ln(\ln x) + \cos x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \left(\frac{\cos x}{x\ln x} - \sin x\ln(\ln x)\right)y$$

$$= \left(\frac{\cos x}{x\ln x} - \sin x\ln(\ln x)\right)(\ln x)^{\cos x}$$

Example 11: Find an equation of the tangent line to $f(x) = \ln(x + \ln x)$ at $x = 1$.

$$f'(x) = \frac{1}{x+\ln x} \cdot \frac{d}{dx}(x + \ln x)$$

$$= \frac{1}{x+\ln x} \cdot \left(1 + 1/x\right)$$

$$m = \frac{1}{1+\ln 1}\left(1 + 1/1\right)$$

$$m = 2$$

$$x = 1, \quad y = f(1) = \ln(1 + \ln 1) = \ln 1 = 0$$

$$\boxed{y - 0 = 2(x-1)} \quad \boxed{y = 2x - 2}$$

Example 12: Let $f(x) = cx + \ln(\sin x)$. For what value of c is $\underline{f'(\pi/4) = 6}$?

$$f'(x) = c + \frac{1}{\sin x} \cdot \cos x$$

$$f'(x) = c + \cos x / \sin x$$

$$6 = c + \frac{\cos \pi/4}{\sin \pi/4}$$

$$6 = c + 1$$

$$\boxed{c = 5}$$